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# Structures of Lyapunov Regular Sets with Non-Zero Exponents

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## 1 Inroduction

Let $M$ be a closed manifold with a Riemannian metric and $f; M \to M$ be a $C^2$-diffeomorphism. For $\lambda, \mu > 0$ and $1 \leq k \leq \dim M - 1$, we denote by $\Lambda = \Lambda(-\mu, \lambda, k)$ the set consisting of Lyapunov regular points $x$ such that

1. Lyapunov exponents of $x$ are less than $-\mu$ or greater than $\lambda$,

2. the dimension of the fiber of stable bundle $E^s(x)$ equals $k$.

Without loss of generality, we may assume that $\Lambda$ has a dense orbit. In this note we construct a "symbolic dynamics "that represents $\Lambda$, and study the structure of $\Lambda$ in the case when $\Lambda$ is a fractal set.

## 2 symbolic dynamics of $\Lambda$

The set $\Lambda$ is represented by a "symbolic dynamics "as follows.
**I** Let $\mathcal{W} = \{W_n\}_{n \geq 0}$ be a family of sets $W_n$ of words of length $n+1$ such that

1. There are symbols $\{B_1, \cdots, B_p, C_1, \cdots, C_q\}$, and for any $n \geq 0$

$$W_n \subset \{B_1, \cdots, B_p, C_1, \cdots, C_q\}^{n+1},$$

2. $(\alpha_0, \cdots, \alpha_n) \in W_n$ implies $\alpha_0, \alpha_n \in \{B_1, \cdots, B_p\}$,

3. If $(\alpha_0, \cdots, \alpha_n) \in W_n, (\beta_0, \cdots, \beta_m) \in W_m$ and $\alpha_n = \beta_0$, then $(\alpha_0, \cdots, \alpha_n, \beta_1, \cdots, \beta_m) \in W_{n+m}$,

4. for any $B_i, B_j$ $(1 \leq i, j \leq p)$, there are $n \geq 1$ and $(\alpha_0, \cdots, \alpha_n) \in W_n$ with $\alpha_0 = B_i, \alpha_n = B_j$.

**II** For a family of sets of words $\mathcal{W} = \{W_n\}_{n \geq 0}$ as above, we define a subset of shift (not necessarily subshift) $\Sigma = \Sigma(\mathcal{W})$ as follows:

1. $\Sigma = \Sigma(\mathcal{W}) \subset \{B_1, \cdots, B_p, C_1, \cdots, C_q\}^{\mathbb{Z}}$

2. $\Sigma$ is generated by $\mathcal{W} = \{\mathcal{W}_n\}_{n \geq 0}$, that is,

   (a) $\underline{\alpha} = (\alpha_n)_{n \in \mathbb{Z}} \in \Sigma$ if and only if for any $N > 0$ there are $m, n \geq N$ such that $(\alpha_{-m}, \alpha_{-m+1}, \cdots, \alpha_0, \cdots, \alpha_n) \in W_{m+n}$

(b) $\alpha_0 \in \{B_0, \cdots, B_p\}$

**III** We denote by $\mathbf{\Sigma}$ a collection of $\Sigma = \Sigma(\mathcal{W})$, where $\mathcal{W} = \{W_n\}_{n \geq 0}$, is defined in **I** and **II**. And an equivalence relation $\sim$ is defined in the disjoint union $\cup\, \Sigma(\mathcal{W})$ as follows:

if $\underline{a} \sim \underline{b}$ for $\underline{a} \in \Sigma, \underline{b} \in \Sigma'$ and $\sigma^n(\underline{a}) \in \Sigma, \sigma^n(\underline{b}) \in \Sigma'$
then $\sigma^n(\underline{a}) \sim \sigma^n(\underline{b})$, where $\sigma$ is the shift map.

**IV** For the quotient space $\tilde{\Sigma} = \cup\, \Sigma(\mathcal{W})/\sim$, a shift map $\tilde{\sigma} : \tilde{\Sigma} \to \tilde{\Sigma}$ is defined as follows:

for $\underline{\alpha} = (\alpha_n)_n \in \Sigma$ with $\alpha_0, \alpha_k \in \{B_1, \cdots, B_p\}$, we have $\tilde{\alpha}^k[\underline{\alpha}] = [\underline{\beta}]$, where $\underline{\beta} = (\beta_n)_n \in \Sigma$ is given by $\beta_n = \alpha_{n+k}$.

With the notations as above, we have the following.

**Theorem 2.1.** *For a Lyapunov regular set $\Lambda = \Lambda(-\mu, \lambda, k)$, there are*

1. *a collection $\mathbf{\Sigma} = \{\Sigma(\mathcal{W})\}$ of countable subsets of shifts $\Sigma(\mathcal{W})$,*
2. *an equivalence relation $\sim$ on $\cup\, \Sigma(\mathcal{W})$ and the shift map $\tilde{\sigma} : \tilde{\Sigma} = \cup\, \Sigma(\mathcal{W})/\sim \,\to \tilde{\Sigma}$,*
3. *a collection of maps $\Psi = \Psi_\Sigma : \Sigma \to \Lambda$ (for $\Sigma \in \mathbf{\Sigma}$ ) which is compatible with the equivalence relation $\sim$,*

*such that the map*

$$\tilde{\Psi} : \tilde{\Sigma} \longrightarrow \Lambda$$

*induced from $\{\Psi = \Psi_\Sigma \mid \Sigma \in \mathbf{\Sigma}\}$ is surjective and the diagram*

$$\begin{array}{ccc} \tilde{\Sigma} & \xrightarrow{\tilde{\sigma}} & \tilde{\Sigma} \\ {\scriptstyle \tilde{\Psi}}\downarrow & & \downarrow{\scriptstyle \tilde{\Psi}} \\ \Lambda & \xrightarrow[f|\Lambda]{} & \Lambda \end{array}$$

*is commutative.*

**Remark 2.1.** *Let $\varepsilon$ be an arbitrary positive number. In Theorem2.1. we may choose the symbols $\{B_1, \cdots, B_p, C_1, \cdots, C_q\}$ of any $\mathcal{W}$ and maps $\Psi = \Psi_\Sigma$ such that*

1. *$p = 1$,*
2. $\operatorname{diam} \Psi(\Sigma) < \varepsilon$ *for $\Sigma = \Sigma(\mathcal{W}) \in \mathbf{\Sigma}$,*
3. *for $x = \Psi((\alpha_n)_n) \in \Psi(\mathcal{W})$*
   $\|\mathrm{T}f^n \mid \mathrm{E}^s(x)\| < \exp(-\mu n)$ *if $\alpha_0 = \alpha_n = B_1$,*
   $\|\mathrm{T}f^{-n} \mid \mathrm{E}^u(x)\| < \exp(-\lambda n)$ *if $\alpha_{-n} = \alpha_0 = B_1$.*

# 3 Locally self-similarity with countable contractions

Let $\Lambda = \Lambda(-\mu, \lambda, k)$ be a Lyapunov regular set. In the sequel we assume that $\mathbf{\Sigma} = \{\Sigma\}$,where $\Sigma = \Sigma(\mathcal{W})$, and maps $\Psi = \Psi_\Sigma : \Sigma \to \Lambda$ $(\Sigma \in \mathbf{\Sigma})$ are given as in section 2 and satisfy Remark2.1.

Then $\Lambda$ is a countable union of closed sets:

$$\Lambda = \bigcup_{\Sigma \in \mathbf{\Sigma}} \Psi(\Sigma).$$

In this section we consider the structure of $\Psi(\Sigma)$.

Let $G_k(\mathcal{W})$ be the set of generators of $\mathcal{W}$, that is,

$$G_k(\mathcal{W}) = \{(a_0, \cdots .a_k) \in W_k \mid a_0 = a_k = B_1, a_i \in \{C_1, \cdots, C_q\}\ 1 \le i \le k-1\},$$
$$G(\mathcal{W}) = \bigcup_k G_k(\mathcal{W}).$$

For $\mathbf{a} = (a_0, \cdots, a_k) \in G_(\mathcal{W})$, the right contraction

$$R(\mathbf{a})\colon \Psi(\Sigma) \longrightarrow \Psi(\Sigma)$$

is defined by

$$R(\mathbf{a})(\Psi((\alpha_n)_n) = \Psi((\beta_n)_n) \quad for\ (\alpha_n)_n \in \Sigma,$$

where

$$\beta_n = \begin{cases} \alpha_{n-k}, & k \le n, \\ a_n, & 0 \le n \le k, \\ \alpha_n, & n \le 0. \end{cases}$$

Similarly the left contraction

$$L(\mathbf{a})\colon \Psi(\Sigma) \longrightarrow \Psi(\Sigma)$$

is defined by

$$L(\mathbf{a})(\Psi((\alpha_n)_n) = \Psi((\beta_n)_n) \quad for\ (\alpha_n)_n \in \Sigma,$$

where

$$\beta_n = \begin{cases} \alpha_n, & 0 \le n, \\ a_{n+k}, & -k \le n \le 0, \\ \alpha_{n+k}, & n \le -k. \end{cases}$$

Then we have the following.

**Proposition 3.1.** *The set $\Psi(\Sigma)$ is a countable union of images of $\Psi(\Sigma)$ by maps $R(\mathbf{a})L(\mathbf{b})$;*

$$\Psi(\Sigma) = \bigcup_{\mathbf{a},\mathbf{b} \in G(\mathcal{W})} R(\mathbf{a})L(\mathbf{b})(\Psi(\Sigma)).$$

*If the map $\Psi = \Psi_\Sigma\colon \Sigma \to \Lambda$ is injective, then for any $\mathbf{a}, \mathbf{b} \in G(\mathcal{W})$ the map*

$$L(\mathbf{b})R(\mathbf{a}) = R(\mathbf{a})L(\mathbf{b})\colon \Psi(\Sigma) \longrightarrow \Sigma(\Sigma)$$

*is a contraction.And $\Psi(\Sigma)$ is self-similar by countable contractions.*

# 4 Hausdorff dimension of local stable manifolds

Form the propositions in the revious section,we have

$$\begin{aligned}
\Psi(\Sigma) &= \bigcup_{\mathbf{a}_1,\mathbf{b}_1\in G(\mathcal{W})} R(\mathbf{a}_1)L(\mathbf{b}_1)\,(\Psi(\Sigma)) \\
&= \bigcup_{\mathbf{a}_2,\mathbf{b}_2\in G(\mathcal{W})} \bigcup_{\mathbf{a}_1,\mathbf{b}_1\in G(\mathcal{W})} R(\mathbf{a}_2)L(\mathbf{b}_2)R(\mathbf{a}_1)L(\mathbf{b}_1)\,(\Psi(\Sigma)) \\
&= \bigcup_{\mathbf{a}_1,\mathbf{a}_2\in G(\mathcal{W})} \bigcup_{\mathbf{b}_1,\mathbf{b}_2\in G(\mathcal{W})} R(\mathbf{a}_2)R(\mathbf{a}_1)L(\mathbf{b}_2)L(\mathbf{b}_1)\,(\Psi(\Sigma)) \\
&= \cdots \\
&= \bigcup_{\underline{\mathbf{a}},\underline{\mathbf{b}}\in G(\mathcal{W})^{\mathbb{N}}} R(\underline{\mathbf{a}})L(\underline{\mathbf{b}})\,(\Psi(\Sigma))\,,
\end{aligned}$$

where

$$\underline{\mathbf{a}} = (\mathbf{a}_1,\mathbf{a}_2,\cdots),\ \underline{\mathbf{b}} = (\mathbf{b}_1,\mathbf{b}_2,\cdots) \in G(\mathcal{W})^{\mathbb{N}},$$

and

$$L(\underline{\mathbf{b}})\,(\Psi(\Sigma)) = \bigcap_{N\geq 1} L(\mathbf{b}_N)\cdots L(\mathbf{b}_1)\,(\Psi(\Sigma))\,.$$

Besides the set $L(\underline{\mathbf{b}})\,(\Psi(\Sigma))$ coincides with an intersection of a local unstable manifold and $\Psi(\Sigma)$.

Let $\mathrm{Lip}\,(R(\mathbf{a}) \mid L(\underline{\mathbf{b}})\,(\Psi(\Sigma)))$ be the Lipshitz constant of the map

$$R(\mathbf{a})\colon L(\underline{\mathbf{b}})\,(\Psi(\Sigma)) \longrightarrow L(\underline{\mathbf{b}})\,(\Psi(\Sigma))\,.$$

By choosing $\varepsilon > 0$ in Remark2.1 sufficiently small, we have

$$\mathrm{Lip}\,(R(\mathbf{a}) \mid L(\underline{\mathbf{b}})\,(\Psi(\Sigma))) < \exp(-\lambda n).$$

Because the number of the elements of $G_k(\mathcal{W})$ is less than or equals $q^{k-1}$, this implies the following.

**Proposition 4.1.** *For* $\underline{\mathbf{b}} \in G(\mathcal{W})^{\mathbb{N}}$*, there is* $c(\underline{\mathbf{b}}) > 0$ *such that*

$$\sum_{\mathbf{a}\in G(\mathcal{W})} \mathrm{Lip}\,(R(\mathbf{a}) \mid L(\underline{\mathbf{b}})\,(\Psi(\Sigma)))^{c(\underline{\mathbf{b}})} = 1.$$

The number $c(\underline{\mathbf{b}})$ dominates the Hausdorff dimension of the intersection of the local unstable manifold and $\Psi(\Sigma)$:

**Proposition 4.2.** *The Hausdorff dimension of* $L(\underline{\mathbf{b}})\,(\Psi(\Sigma))$ *is less than or equals* $c(\underline{\mathbf{b}})$.

# References


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